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Numerical Solution of Integral Equations of Mathematical Physics, Using Chebyshev Polynomials

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We present a method for numerically solving certain linear Fredholm integral equations of the second kind. The solution involves a Chebyshev series approximation, the coefficients of which are solutions of a linear system of equations. The coefficients of these equations are computed using recurrence relations that depend on the kernel. For a number of important kernels, the recurrence relations are given in this paper. To illustrate the method, it is applied to an integral equation that arises in the theory of intrinsic viscosity of macromolecules and to Love's integral equation. Both equations have been dealt with elsewhere, and we compare our results to those already published.

1. INTRODUCTION

In this paper we consider a numerical method for the solution of the Fredholm integral equation of the second kind

$$\phi(x) = f(x) - \lambda \int_{-1}^{+1} K(x, y)\, \phi(y)\, dy, \tag{1}$$

where ϕ is the function to be determined. The constant λ, the kernel function K and the function f are given. Since by means of a linear transformation, any finite interval $[a, b]$ can be converted to $[-1, 1]$, we assume that the range of the variables is normalized so that $-1 \leqslant x, y \leqslant 1$.

The solution of (1) is expressed as a series

$$\phi(x) = w(x) \sum_{k=0}^{\infty} c_k T_k(x), \tag{2}$$

where $T_k(x)$ is the Chebyshev polynomial of the first kind and of degree k, and where the function w has to be chosen so that the Chebyshev series expansion in the right-hand member of (2), is rapidly converging. This means, that w has to contain the same singularities as ϕ. In many cases, $w \equiv 1$ is a convenient choice. Of course, for numerical computations, the infinite series (2) has to be truncated after a finite number of terms, say $N + 1$, so that

$$\phi(x) = w(x) \sum_{k=0}^{N} c_k T_k(x) + \epsilon(x), \tag{3}$$

where $\epsilon(x)$ is the remainder.

It is our purpose to determine the coefficients $c_0, c_1, \ldots, c_N$. The use of the Chebyshev series for the numerical solution of linear integral equations has previously been discussed by Elliott [1, 2], Fox and Parker [3], and Scraton [4], and of nonlinear integral equations by Wolfe [5], Sag [6], and Shimasaki and Kiyono [7]. All methods are based on the same idea. Substituting (3) in (1) yields a system of linear or nonlinear equations in the unknown coefficients $c_0, c_1, \ldots, c_N$. They differ only in the method of setting up this system of equations and in the method of solving it. In Elliott's method, the kernel function $K(x_i, y)$ is approximated by a polynomial of degree M in the form

$$K(x_i, y) = \sum_{n=0}^{M} b_n(x_i)\, T_n(y), \tag{4}$$

for $N + 1$ different values x_i, $i = 0, 1, \ldots, N$, and then the system

$$\sum_{k=0}^{N} c_k T_k(x_i) = f(x_i) - \lambda \int_{-1}^{+1} \left[\sum_{n=0}^{M} b_n(x_i)\, T_n(y) \right] \sum_{k=0}^{N} c_k T_k(y)\, dy, \tag{5}$$

$i = 0, 1, \ldots, N$ has to be solved for the unknowns c_k. The numerical computation of $b_n(x_i)$, $n = 0, 1, \ldots, M$ and $i = 0, 1, \ldots, N$ requires $(M + 1)(N + 1)$ evaluations of $K(x, y)$. Moreover, for each value of i, the number of multiplications is of order $O(M^2)$ if Clenshaw's algorithm [8] is used or of order $O(M \log M)$, if the FFT-algorithm is used [9]. Especially if K is a singular kernel or a strongly oscillating or peaked function, the value of M has to be chosen very large, in order to have a reasonable accuracy. Elliott's method (and also the other methods) requires then a large amount of computational effort and often it is even not practically applicable. We present here a new, efficient method using recurrence formulas, which is, however, not generally applicable. Nevertheless, it is applicable for some important integral equations of mathematical physics.

2. Method of Solution

Substituting (3) into (1) and neglecting $\epsilon(x)$, we have

$$\sum_{k=0}^{N} c_k [w(x)\, T_k(x) + \lambda I_k(x)] = f(x), \tag{6}$$

where

$$I_k(x) = \int_{-1}^{+1} K(x, y)\, w(y)\, T_k(y)\, dy. \tag{7}$$

Substituting $N + 1$ values of x, say $x_0, x_1, \ldots, x_N$, into (6), we obtain a system of linear equations, the solution of which gives approximate values of the Chebyshev series coefficients c_k. In our following computations, we choose the abscissas x_k equidistantly between -1 and 1, or, if ± 1 are singular points, between -0.95 and $+0.95$. Of course, more than $N + 1$ values of x can be substituted, but then we have to determine a solution of an overdetermined system of equations.

The remaining problem is the evaluation of $I_k(x_j)$ for $j, k = 0, 1, \ldots, N$. In a number of important cases, this can be done using linear recurrence relations, as we shall illustrate in the following sections. The derivation of these recurrence relations is based on the properties of the Chebyshev polynomials [3]. We give one completely developed example in the Appendix. The other recurrence relations in this paper can be derived in an analogous way.

Of course, since recurrence relations are susceptible to error growth, we have to be concerned with the numerical stability. A very interesting review of the computational aspects of homogeneous three-term recurrence relations is given by Gautschi [10]. More general recurrence relations are studied by Oliver [11, 12]. This author gives an algorithm for the stable computation of the solutions of recurrence relations, if forward recursion is unstable. The essential idea of this algorithm, in the case of a $(n + 1)$-term recurrence relation, is the replacement of the n initial conditions (starting values) by k initial conditions and $n - k$ end conditions. The value of k depends on the relative behavior of all solutions of the recurrence relation. Details of our study of the numerical stability of the recurrence relations considered in this paper, will appear in a later publication. Here we shall only indicate whether forward recursion is stable or Oliver's algorithm has to be used.

The computations in the following sections were carried out on a IBM 370/158 computer, using double precision arithmetic. All computer programs were written in FORTRAN IV and the compiler used was the level G-compiler. For the solution of the linear system of equations the program of Forsythe and Moler [13] was used.

3. Approximate Solution of a Singular Integral Equation Arising in Polymer Physics

The theory of intrinsic viscosity, developed by Kirkwood and Riseman [14] requires the solution of the linear weakly singular integral equation

$$\phi(x) = f(x) - \lambda \int_{-1}^{+1} \phi(y)\, |\, x - y\,|^{-\alpha}\, dy, \qquad 0 < \alpha < 1, \tag{8}$$

The solutions of this equation have been discussed by Auer and Gardner [15] and numerical methods of solution have been presented by Ullman [16, 17], Schlitt [18], and more recently by Cohen and Ickovic [19]. These methods have the disadvantage that the solution $\phi(x)$ is computed at a restricted number of values of x determined by some type of quadrature formula. To find $\phi(x)$ at other values of x an interpolation scheme can be used or all computations have to be repeated using another quadrature formula.

We assume that f is an even function, but the general case can be solved in the same way. Note that ϕ then also is an even function. Assume that the solution can be represented by

$$\phi(x) = \sum_{k=0}^{[N/2]} c_{2k} T_{2k}(x), \tag{9}$$

where $[N/2]$ means the integer part of $N/2$.

To construct the system of linear equations for the determination of the c_{2k}, we have to evaluate

$$I_k(x) = \int_{-1}^{+1} | x - y |^{-\alpha} T_k(y)\, dy, \qquad k = 0, 1, 2,..., N.$$

The recurrence relation for $I_k(x)$ is, for $| x | < 1$,

$$\left[1 + \frac{1 - \alpha}{k + 1}\right] I_{k+1}(x) - 2xI_k(x) + \left[1 - \frac{1 - \alpha}{k - 1}\right] I_{k-1}(x)$$
$$= \frac{2}{1 - k^2} [(1 - x)^{1-\alpha} - (-1)^k (1 + x)^{1-\alpha}], \tag{10}$$

which can be solved by forward recursion in a numerically stable way.

Starting values for this recurrence relation are,

$$I_0(x) = (1/(1 - \alpha))[(x + 1)^{1-\alpha} + (1 - x)^{1-\alpha}]$$
$$I_1(x) = xI_0(x) + (1/(2 - \alpha))[(1 - x)^{2-\alpha} - (1 + x)^{2-\alpha}]$$

and

$$I_2(x) = 4xI_1(x) - (2x^2 + 1)\, I_0(x) + (2/(3 - \alpha))[(1 - x)^{3-\alpha} + (1 + x)^{3-\alpha}]. \tag{11}$$

To evaluate the usefulness of our method and to allow comparison with the published results in [17–19] we have solved (8) for the case where $f(x) = x^2$, $\alpha = \frac{1}{2}$ and $\lambda = 0.5$, 5, 20, and 200.

In Table I, we compare some numerical values of our solution ($N = 20$) with those obtained by other methods, which use 20-point quadrature formulas. In Table II, we present the coefficients c_{2k}. As can be seen, all methods agree with

TABLE I

Values of ϕ in Selected Abscissas of the 20-point Gauss–Legendre Quadrature Formula[a]

λ	x	Ullman	Schlitt	Cohen–Ickovic	Present method
0.5	0.99313	0.63303	0.63130	0.62997	0.62856
	0.96397	0.55114	0.54817	0.54650	0.55111
	0.74633	0.25311	0.25121	0.24995	0.25121
	0.51087	0.06767	0.06737	0.06631	0.06733
	0.07653	−0.07907	−0.07790	−0.07882	−0.07799
5.0	0.99313	0.19322	0.19148	0.19018	0.18659
	0.96397	0.13553	0.13346	0.13112	0.13789
	0.74633	0.04332	0.04269	0.04209	0.04263
	0.51087	0.00358	0.00369	0.00332	0.00364
	0.07653	−0.02496	−0.02429	−0.02456	−0.02434
20	0.99313	0.06127	0.06049	0.05998	0.05807
	0.96397	0.03873	0.03803	0.02695	0.04007
	0.74633	0.01135	0.01174	0.01099	0.01114
	0.51087	0.00049	0.00054	0.00044	0.00052
	0.07653	−0.00718	−0.00697	−0.00704	−0.00699
200	0.99313	0.00671	0.00661	0.00655	0.00630
	0.96397	0.00404	0.00397	0.00382	0.00422
	0.74633	0.00115	0.00113	0.00111	0.00113
	0.51087	0.00003	0.00004	0.00003	0.00004
	0.07653	−0.00075	−0.00073	−0.00074	−0.00073

[a] Obtained by the methods of Ullman, Schlitt, Cohen, and Ickovic; and our method.

each other quite well and it is difficult to say which method is the more accurate.

Auer and Gardner [15] have obtained an asymptotic solution of (8). For $f(x) = x^2$ and $\alpha = \frac{1}{2}$ they obtained

$$\phi(x) \sim \sqrt{2}(4x^2 - 1)(1 - x^2)^{-1/4}/3\pi\lambda, \qquad \lambda \to \infty. \tag{12}$$

Thus, for $\lambda \to \infty$, the solution of (8) has singularities at $x = \pm 1$. Therefore

TABLE II

Approximate Values of the Chebyshev Coefficients c_{2k} of the Solution of

$$\phi(x) = x^2 - \lambda \int_{-1}^{+1} \phi(y) \, |x - y|^{-1/2} \, dy$$

k	$\lambda = 0.5$	$\lambda = 5$	$\lambda = 20$	$\lambda = 200$
0	0.486380	0.108225	0.031321	0.0033030
1	0.349607	0.096094	0.028691	0.0030568
2	0.034159	0.025573	0.009097	0.0010305
3	0.013101	0.013312	0.005124	0.0005975
4	0.006146	0.007348	0.003003	0.0003574
5	0.002943	0.003951	0.001668	0.0002016
6	0.001313	0.001904	0.000828	0.0001013
7	0.000507	0.000778	0.000347	0.0000428
8	0.000156	0.000250	0.000113	0.0000141
9	0.000033	0.000055	0.000026	0.0000032
10	0.000004	0.000006	0.000003	0.0000004

for large values of λ, we may expect that, instead of (9), the solution of (8) can be represented more accurately by

$$\phi(x) = (1 - x)^{\beta} (1 + x)^{\gamma} \sum_{k=0}^{[N/2]} d_{2k} T_{2k}(x), \tag{13}$$

where the best values of β and γ are probably $-\frac{1}{4}$.

To calculate d_{2k} in the way described above, we have to evaluate

$$J_k(x) = \int_{-1}^{+1} |x - y|^{-\alpha} (1 - y)^{\beta} (1 + y)^{\gamma} T_n(y) \, dy, \tag{14}$$

for $\alpha = \frac{1}{2}$ and $\beta = \gamma = -\frac{1}{4}$. This can be done by the recurrence relation

$$\begin{aligned}(\beta + \gamma - \alpha + 3 + k) J_{k+2}(x) &+ 2[(1 - x)(1 + \beta) - (1 + x)(1 + \gamma) - xk] J_{k+1}(x) \\ &+ 2[(1 - 2x)(1 + \beta) + (1 + 2x)(1 + \gamma) + \alpha - 1] J_k(x) \\ &+ 2[(1 - x)(1 + \beta) - (1 + x)(1 + \gamma) + xk] J_{k-1}(x) \\ &+ (\beta + \gamma - \alpha + 3 - k) J_{k-2}(x) = 0.\end{aligned} \tag{15}$$

with starting values

$$\begin{aligned} J_0(x) &= G(\alpha, \beta, \gamma, x), \\ J_1(x) &= G(\alpha, \beta, \gamma + 1, x) - J_0(x), \end{aligned} \tag{16}$$

where

$$\begin{aligned} G(\alpha, \beta, \gamma, x) = {} & 2^\beta (1 + x)^{\gamma - \alpha + 1} B(\gamma + 1, 1 - \alpha) \\ & \times {}_2F_1(-\beta, \gamma + 1; 2 + \gamma - \alpha; (1 + x)/2) \\ & + 2^\gamma (1 - x)^{\beta - \alpha + 1} B(\beta + 1, 1 - \alpha) \\ & \times {}_2F_1(-\gamma, \beta + 1; 2 + \beta - \alpha; (1 - x)/2), \end{aligned}$$

where $B(\alpha, \beta) = \Gamma(\alpha) \Gamma(\beta)/\Gamma(\alpha + \beta)$ is the beta function and ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

Relation (15) remains valid for $k = 0$ and 1, if $J_{k-1}(x)$ and $J_{k-2}(x)$ are replaced by $J_{|k-1|}(x)$ and $J_{|k-2|}(x)$. So, if $J_0(x)$ and $J_1(x)$ are known, $J_2(x)$ and $J_3(x)$ can be computed, using (15), and forward recursion, which is numerically stable, can be started.

Although the expressions for $J_k(x)$, $k = 0, 1$ are suitable for numerical computations, it seems easier and more efficient to evaluate $J_k(x)$ by numerical calculation of the integrals in the right-hand member of

$$\begin{aligned} J_k(x) = {} & \int_{-1}^{x} (x - y)^{-\alpha} (1 + y)^\gamma (1 - y)^\beta T_k(y)\, dy \\ & + \int_{x}^{1} (y - x)^{-\alpha} (1 - y)^\beta (1 + y)^\gamma T_k(y)\, dy, \end{aligned} \tag{17}$$

using the automatic integrator AINAB, presented in [20] which is tailored for these types of integrals. Moreover, if $\beta = \gamma$ and $2\beta - \alpha + 1 = 0$, it can be shown that

$$J_0(x) = -\pi/\sin \beta\pi, \qquad \text{(independent of } x\text{)},$$

and

$$J_1(x) = \alpha x J_0(x).$$

Thus, in our case where $\alpha = \frac{1}{2}$ and $\beta = \gamma = -\frac{1}{4}$, we have

$$\begin{aligned} J_0(x) &= \sqrt{2}\pi, \\ J_1(x) &= x\pi/\sqrt{2}, \\ J_2(x) &= (3x^2 - 2)\, \pi/2^{3/2}, \\ J_3(x) &= (5x^3 - 4x)\, \pi/2^{3/2}. \end{aligned} \tag{18}$$

TABLE III

Approximate Solution of Eq. (8) Computed using the Expression (13)

λ	x	$N = 20$	$N = 24$	$N = 30$
0.5	0.99313	0.734579	0.717451	0.701180
	0.96397	0.546128	0.545532	0.545315
	0.74633	0.249856	0.250040	0.250216
	0.51087	0.066419	0.066541	0.066658
	0.07653	−0.078712	−0.078616	−0.078524
5	0.99313	0.201681	0.199768	0.197854
	0.96397	0.131435	0.131519	0.131652
	0.74633	0.042418	0.042444	0.042469
	0.51087	0.003528	0.003542	0.003556
	0.07653	−0.024427	−0.024417	−0.024407
20	0.99313	0.060277	0.060092	0.059902
	0.96397	0.037604	0.037617	0.037636
	0.74633	0.011129	0.011132	0.011134
	0.51087	0.000513	0.000515	0.000516
	0.07653	−0.006995	−0.006995	−0.006994
200	0.99313	0.00641311	0.00641096	0.00640874
	0.96397	0.00393224	0.00393241	0.00393266
	0.74633	0.00112775	0.00112778	0.00112781
	0.51087	0.00003723	0.00003724	0.00003726
	0.07653	−0.00073023	−0.00073022	−0.00073021

In Table III, we give the results for $\lambda = 0.5, 5, 20$, and 200, computed with $N = 20$, 24, and 30.

In Table IV, the coefficients d_{2k}, $k = 0, 1, \ldots, 12$ are reported (coefficients which are less than 10^{-7} are omitted). The values of d_{2k} as well as the results in Table III show that (13) is a good approximation, which converges satisfactorily for increasing values of N, especially if λ is large. The reliability of the results of Table III was checked by recomputing them using another set of x_k-values, so that another system of equations for the coefficients d_{2k} was to be solved. These check calculations lead to the conclusion that the values in Table III for $N = 30$ are accurate to at least two and often even three significant figures.

TABLE IV

Coefficients d_{2k} of the Approximate Solution $\phi(x) = (1 - x^2)^{-1/4} \sum_{k=0}^{[N/2]} d_{2k} T_{2k}(x)$ of the Integral Eq. (8)

k	$\lambda = 0.5$	$\lambda = 5$	$\lambda = 20$	$\lambda = 200$
0	0.271389	0.051076	0.014418	0.0014943
1	0.180131	0.048666	0.014171	0.0014917
2	−0.051883	−0.003590	−0.000306	−0.0000034
3	−0.021577	−0.001987	−0.000181	−0.0000020
4	−0.010586	−0.001151	−0.000110	−0.0000013
5	−0.005388	−0.000656	−0.000065	−0.0000008
6	−0.002708	−0.000359	−0.000037	−0.0000004
7	−0.001304	−0.000184	−0.000019	−0.0000002
8	−0.000588	−0.000087	−0.000009	
9	−0.000242	−0.000038	−0.000004	
10	−0.000089	−0.000014	−0.000002	
11	−0.000028	−0.000005		
12	−0.000007	−0.000001		

Since the applications of the solutions of (8) to polymer physics involve an integral of $\phi(x)$ over x, another advantage of our method is that (9) can be integrated analytically [3] and that the integral of (13) can be computed accurately using Gauss–Jacobi quadrature formulas [21], or using the integrator AINAB [20].

4. Love's Integral Equation

In [22], Love shows that the field of two equal circular coaxial conducting disks of radius r, separated by a distance $a \times r$ and at equal or opposite potential, with zero potential at infinity, is given by the solution of the integral equation

$$\phi(x) = 1 + \lambda \int_{-1}^{+1} \frac{a}{a^2 + (x - y)^2} \phi(y) \, dy, \tag{19}$$

where $\lambda = \pm 1/\pi$.

This equation with $a = 1$ has been considered by numerous authors. It is treated by Fox and Goodwin [23], using finite-difference methods, by Brakhage

[24] and Boland [25] using a quadrature formula method and by Elliott [2] and Fox and Parker [3] using Chebyshev series approximations.

Our method requires the evaluation of

$$I_k(x) = \int_{-1}^{+1} \frac{a}{a^2 + (x - y)^2} T_k(y)\, dy \tag{20}$$

using the recurrence relation

$$\begin{aligned} &I_{k+2}(x) - 4xI_{k+1}(x) + (2 + 4a^2 + 4x^2)\, I_k(x) - 4xI_{k-1}(x) \\ &+ I_{k-2}(x) = (4a/(1 - k^2))[1 + (-1)^k], \end{aligned} \tag{21}$$

where, if $a \neq 0$,

$$\begin{aligned} I_0(x) &= \arctan \frac{1 - x}{a} + \arctan \frac{1 + x}{a}, \\ I_1(x) &= xI_0(x) + \frac{a}{2} \ln \frac{(1 - x)^2 + a^2}{(1 + x)^2 + a^2}, \\ I_2(x) &= 4xI_1(x) - (2a^2 + 2x^2 + 1)\, I_0(x) + 4a, \\ I_3(x) &= -(4a^2 - 12x^2 + 3)\, I_1(x) - 8x(a^2 + x^2)\, I_0(x) + 16xa. \end{aligned} \tag{22}$$

Since the kernel of (19) is symmetric and infinitely differentiable the solution can be represented by a rapidly converging Chebyshev series of the form (9).

TABLE V

The Coefficients c_{2k} for Love's Equation $\phi(x) = \sum_{k=0}^{9} c_{2k} T_{2k}(x)$

k	$\lambda = -1/\pi$	$\lambda = 1/\pi$
0	3.5488896186	1.4151851841
1	−0.1400430542	0.0493850582
2	0.0049619937	−0.0010475174
3	0.0003762979	−0.0002327553
4	−0.0000436869	0.0000199861
5	−0.0000016232	0.0000009868
6	0.0000004965	−0.0000002380
7	−0.0000000064	0.0000000019
8	−0.0000000051	0.0000000024
9	0.0000000003	0.0000000001

In Table V, the coefficients c_{2k} are reported. They agree to within six to seven decimal figures with the numbers given by Elliott [2]. Our solution is accurate to 10 decimal places. This was checked by recomputing it with more terms in the Chebyshev series approximation, and using extended precision arithmetic.

The numerical instability of forward recursion of (21) increases with increasing values of $|a|$. On the other hand, the Chebyshev series expansion of the solution is faster converging, such that less terms are needed to obtain a given accuracy. It turns out that, if the requested accuracy of the solution is not higher than 10^{-10} and if double precision arithmetic is used, forward recursion may be applied.

5. Other Recurrence Relations

If the kernel of the Fredholm integral equation (1) is equal to or can be approximated by a linear combination of the following functions: $|x-y|^\alpha$, $|x-y|^\alpha(1-y)^\beta(1+y)^\gamma$, $[a^2+(x-y)^2]^{-1}$, $(1+y)^\alpha \exp[-x(y+1)]$, $\exp(-axy^2)$, $\exp[-ax(y+1)^2]$, $\cos(axy)$, $\sin(axy)$, $|x-y|^\alpha \operatorname{sign}(x-y)$, $(a^2+x^2+y^2)^\beta$, $\exp[-x/(y+1)]$, $(y-x)^{-1}$, where α, β, γ, and a are parameters, the method of solution described in this paper can be applied, using (10), (15), and (21), and the recurrence relations, presented in this section. Moreover, if in these functions x is replaced by an arbitrary function of x, the present method remains applicable. As can be seen all functions are singular, or strongly oscillating or peaked (at least for some values of the parameters α and a), Consequently, from the viewpoint of the amount of computational effort, in most cases the present method will be superior over Elliott's method, even if the starting values of the recurrence relations involve special functions, which are rather difficult to compute.

(i) $I_n(x) = \int_{-1}^{+1} (1+y)^\alpha \exp[-x(y+1)]\, T_n(y)\, dy, \qquad \alpha > -1.$

The recurrence relation is

$$-\frac{x}{4(n+1)} I_{n+2}(x) + \left[\frac{1}{2} + \frac{\alpha+1-x}{2(n+1)}\right] I_{n+1}(x) + \left[1 + \frac{x}{2(n^2-1)}\right] I_n(x)$$
$$+ \left[\frac{1}{2} - \frac{\alpha+1-x}{2(n-1)}\right] I_{n-1}(x) + \frac{x}{4(n-1)} I_{n-2}(x) = -\frac{2^{\alpha+1}e^{-2x}}{n^2-1}. \tag{23}$$

Starting values are

$$I_0(x) = f_1,$$
$$I_1(x) = f_2 - f_1,$$
$$I_2(x) = 2f_3 - 4f_2 + f_1,$$
$$I_3(x) = 4f_4 - 12f_3 + 9f_2 - f_1,$$

where

$$f_n = P(\alpha + n, 2x)\, x^{-(\alpha+n)}$$

where

$$P(a, x) = \int_0^x e^{-t} t^{a-1}\, dt.$$

$P(a, x)$ can be computed efficiently using the integrator AINAB, given in [20] but if $x > 0$, $P(a, x)/\Gamma(a)$ is the incomplete gamma function, for the computation of which a FORTRAN-program can be found in [26].

For the computation of (23), Oliver's algorithm has to be used with two initial conditions. As a special case we have

$$I_n^*(x) = \int_{-1}^{+1} \exp[-x(y + 1)]\, T_n(y)\, dy.$$

The recurrence relation is

$$\frac{x}{n+1} I_{n+1}^*(x) - 2I_n^*(x) - \frac{x}{n-1} I_{n-1}^*(x) = \frac{2}{n^2 - 1} [\exp(-2x) + (-1)^n] \quad (24)$$

with starting values, for $x \neq 0$,

$$\begin{aligned} I_0^*(x) &= [1 - \exp(-2x)]/x, \\ I_1^*(x) &= (x^{-2} - x^{-1}) - \exp(-2x)(x^{-2} + x^{-1}), \\ I_2^*(x) &= (4x^{-3} - 4x^{-2} + x^{-1}) - \exp(-2x)(4x^{-3} + 4x^{-2} + x^{-1}). \end{aligned}$$

To have numerical stability, Oliver's algorithm has to be used with one initial condition.

(ii) $I_n(x) = \int_{-1}^{+1} \exp(-axy^2)\, T_n(y)\, dy,$

where a is a positive parameter, and $x \neq 0$.

$$I_n(x) = 0, \qquad \text{if } n \text{ is odd.}$$

The recurrence relation is

$$a(n - 1)\, I_{n+2}(x) + 2x^{-1}(1 - n^2 - ax)\, I_n(x) - a(n + 1)\, I_{n-2}(x) = 4x^{-1} \exp(-ax) \quad (25)$$

with starting values

$$I_0(x) = \frac{2}{(ax)^{1/2}} \int_0^{(ax)^{1/2}} \exp(-y^2)\, dy = \left(\frac{\pi}{ax}\right)^{1/2} \operatorname{erf}(ax)^{1/2}, \qquad \text{if} \quad x > 0, \quad (26)$$

or

$$I_0(x) = \frac{2}{(-ax)^{1/2}} \int_0^{(-ax)^{1/2}} \exp(y^2)\, dy, \qquad \text{if} \quad x < 0, \tag{27}$$

and

$$I_2(x) = -2\exp(-ax)/(ax) + (1/(ax) - 1)\, I_0(x).$$

A Chebyshev series approximation for the computation of the integral in (27), which is Dawson's integral, is given in [27].

For (25), Oliver's algorithm has to be used with one initial condition.

(iii) $I_n(x) = \int_{-1}^{+1} \exp[-ax(y+1)^2]\, T_n(y)\, dy,$

where a is a positive parameter, and $x \neq 0$.

The recurrence relation is

$$\begin{aligned} a(n-1)[I_{n+2}(x) + 2I_{n+1}(x)] - (2/x)(ax + n^2 - 1)\, I_n(x) \\ - a(n+1)[2I_{n-1}(x) + I_{n-2}(x)] = (2/x)[\exp(-4ax) + (-1)^n], \end{aligned} \tag{28}$$

with starting values

$$I_0(x) = \frac{1}{2}\left(\frac{\pi}{ax}\right)^{1/2} \operatorname{erf}(2(ax)^{1/2}), \qquad \text{if} \quad x > 0, \tag{29}$$

or

$$I_0(x) = \frac{1}{(-ax)^{1/2}} \int_0^{2(-ax)^{1/2}} \exp(y^2)\, dy, \qquad \text{if} \quad x < 0, \tag{30}$$

and

$$\begin{aligned} I_1(x) &= (2ax)^{-1}[1 - \exp(-4ax)] - I_0(x), \\ I_2(x) &= [1 + (ax)^{-1}]\, I_0(x) - 2(ax)^{-1}, \\ I_3(x) &= -[1 + 6(ax)^{-1}]\, I_0(x) - [2(ax)^{-2} + (2ax)^{-1}] \exp(-4ax) \\ &\quad + 2(ax)^{-2} + 9(2ax)^{-1}. \end{aligned}$$

The integral in (30) is Dawson's integral.

Recurrence relation (28) has to be solved using Oliver's algorithm with 2 initial conditions.

(iv) $I_n(x) = \int_{-1}^{+1} \cos(axy)\, T_n(y)\, dy, \qquad \text{where} \quad x \neq 0.$

The recurrence relation is

$$\begin{aligned} &a^2x^2(n-1)(n-2)\, I_{n+2}(x) - 2(n^2-4)(a^2x^2-2n^2+2)\, I_n(x) \\ &\quad + a^2x^2(n+1)(n+2)\, I_{n-2}(x) = 24ax \sin ax - 8(n^2-4)\cos ax. \end{aligned} \tag{31}$$

Starting values are

$$\begin{aligned} I_0(x) &= 2 \sin ax/(ax), \\ I_2(x) &= 8 \cos ax/(ax)^2 + (2a^2x^2 - 8) \sin ax/(ax)^3, \\ I_4(x) &= 32(a^2x^2 - 12) \cos ax/(ax)^4 + 2(a^4x^4 - 80a^2x^2 + 192) \sin ax/(ax)^5. \end{aligned}$$

Forward recursion of (31) is numerically stable up to $n = [|\, ax \,|]$. For larger values of n, Oliver's algorithm has to be used with one initial condition.

(v) $I_n(x) = \int_{-1}^{+1} \sin(axy)\, T_n(y)\, dy$, where $x \neq 0$

$I_n(x) = 0$, for even values of n.

The recurrence relation is

$$\begin{aligned} &a^2x^2(n-1)(n-2)\, I_{n+2}(x) - 2(n^2-4)(a^2x^2-2n^2+2)\, I_n(x) \\ &\quad + a^2x^2(n+1)(n+2)\, I_{n-2}(x) = -8(n^2-4)\sin ax - 24ax \cos ax. \end{aligned} \tag{32}$$

Starting values are

$$\begin{aligned} I_1(x) &= 2(\sin ax - ax \cos ax)/(ax)^2 \\ I_3(x) &= \sin ax(18 - 48/(ax)^2)/(ax)^2 + \cos ax(48/(ax)^2 - 2)/(ax). \end{aligned}$$

Relation (32) has the same stability properties as (31).

(vi) $I_n(x) = \int_{-1}^{+1} |\, y - x \,|^{\alpha} \operatorname{sign}(y - x)\, T_n(y)\, dy$, $\alpha > -1$.

The recurrence relation is

$$\begin{aligned} &\left(1 + \frac{\alpha+1}{n+1}\right) I_{n+1}(x) - 2xI_n(x) + \left(1 - \frac{\alpha+1}{n-1}\right) I_{n-1}(x) \\ &\quad = \frac{2}{1-n^2}\left[(1-x)^{\alpha+1} + (-1)^n (1+x)^{\alpha+1}\right], \end{aligned} \tag{33}$$

with starting values, if $|x| < 1$,

$$I_0(x) = [(1-x)^{\alpha+1} - (1+x)^{\alpha+1}]/(\alpha+1),$$
$$I_1(x) = xI_0(x) + [(1-x)^{\alpha+2} + (1+x)^{\alpha+2}]/(\alpha+2),$$
$$I_2(x) = 4xI_1(x) - (2x^2+1)\,I_0(x) + 2[(1-x)^{\alpha+3} - (1+x)^{\alpha+3}]/(\alpha+3).$$

Forward recursion is stable.

(vii) $I_n(x) = \int_{-1}^{+1} (a^2 + x^2 + y^2)^{\beta}\, T_n(y)\, dy, \qquad \text{where} \quad a \neq 0,$

$I_n(x) = 0, \qquad$ for odd values of n.

The recurrence relation is

$$\left[\frac{1}{4} + \frac{\beta+1}{2(n+1)}\right] I_{n+2}(x) + \left[\frac{1}{2} + a^2 + x^2 - \frac{\beta+1}{n^2-1}\right] I_n(x)$$
$$+ \left[\frac{1}{4} - \frac{\beta+1}{2(n-1)}\right] I_{n-2}(x) = -2(1+a^2+x^2)^{\beta+1}/(n^2-1). \qquad (34)$$

Starting values are:

For $\beta = -1/2$, $\quad I_0(x) = \ln \dfrac{(1+a^2+x^2)^{1/2}+1}{(1+a^2+x^2)^{1/2}-1}.$

For $\beta = -1$, $\quad I_0(x) = 2(a^2+x^2)^{-1/2} \arctan(a^2+x^2)^{-1/2}.$

For $\beta = -3/2$, $\quad I_0(x) = 2(a^2+x^2)^{-1}(1+a^2+x^2)^{-1/2}.$

For $\beta = -2$, $\quad I_0(x) = (a^2+x^2)^{-1}[(1+a^2+x^2)^{-1} + (a^2+x^2)^{-1/2} \arctan(a^2+x^2)^{-1/2}].$

The value of $I_2(x)$ can be computed from the corresponding value of $I_0(x)$ using the relation

$$I_2(x) = [4(1+a^2+x^2)^{\beta+1} - (3+2\beta+2a^2+2x^2)\, I_0(x)]/(3+2\beta), \qquad \text{if} \quad \beta \neq -3/2,$$

or

$$I_2(x) = 2 \ln \frac{(1+a^2+x^2)^{1/2}+1}{(1+a^2+x^2)^{1/2}-1} - (1+2a^2+2x^2)\, I_0(x), \qquad \text{if} \quad \beta = -3/2.$$

Oliver's algorithm has to be used with one initial condition.

(viii) $I_n(x) = \int_{-1}^{+1} \exp[-x/(y+1)]\, T_n(y)\, dy.$

The recurrence relation is

$$(n+2)(n-1)\, I_{n+1}(x) + [3n^2 + (2x-1)\, n - 2x - 4]\, I_n(x) \\ + [3n^2 - (2x+5)\, n - 2]\, I_{n-1}(x) + n(n-3)\, I_{n-2}(x) = -8 \exp(-x/2). \quad (35)$$

Starting values are, if $x \neq 0$

$$I_0(x) = 2\exp(-x/2) - xE_1(x/2),$$
$$I_1(x) = (x + x^2/2)\, E_1(x/2) - x\exp(-x/2),$$
$$I_2(x) = -(x + 2x^2 + x^3/3)\, E_1(x/2) + 2(x^2 + 4x - 1)\exp(-x/2)/3,$$

where

$$E_1(x) = \int_x^\infty (e^{-t}/t)\, dt, \quad (36)$$

is the exponential integral.

If $x < 0$, the principal value of (36) has to be considered. Chebyshev series approximations for the computation of (36) for both $x > 0$ and $x < 0$ are given in [27].

Forward recursion of (35) is not stable, but the error growth is not disastrous. If high accuracy is required, Oliver's algorithm has to be used with 2 initial conditions.

(ix) $I_n(x) = \int_{-1}^{+1} T_n(x)/(y-x)\, dy.$

The recurrence relation is

$$\begin{aligned} I_{n+1}(x) - 2xI_n(x) + I_{n-1}(x) &= 0, && \text{for } n \text{ odd}, \\ I_{n+1}(x) - 2xI_n(x) + I_{n-1}(x) &= 4/(1-n^2), && \text{for } n \text{ even}. \end{aligned} \quad (37)$$

Starting values are, if $|x| \neq 1$

$$I_0(x) = \ln|(1-x)/(1+x)|$$
$$I_1(x) = 2 + xI_0(x).$$

Forward recursion is stable.

It is important to note that the computation of the starting values of the recurrence relations (23), (24), (28), (31)–(33) is affected by a considerable loss of accuracy if $|x|$ is small. This can be avoided by expanding the expressions for $I_k(x)$ into a Maclaurin series.

6. Conclusion

In this paper, the usefulness of Chebyshev polynomial expansions for the solution of Fredholm integral equations of the second kind is demonstrated. Two integral equations, namely, a singular equation encountered in polymer physics and the nonsingular Love's equation, are chosen to show the accuracy of the method. A considerable advantage of the method is that the solution is expressed as a truncated Chebyshev series. This means that, after calculation of the series coefficients, the solution $\phi(x)$ of the equation can be evaluated for arbitrary values of x at low computation effort. This implies that our method is very efficient. A disadvantage is that the method is not generally applicable, since it requires a recurrence relation which depends on the kernel of the integral equation. However, for some important kernels, the recurrence relations are given in this paper.

Appendix: Construction of Recurrence Relations

The construction of the recurrence relations given in this paper requires a good knowledge of the properties of the Chebyshev polynomials. It is not possible to give generally applicable instructions. We give here one example, namely the derivation of the recurrence relation of

$$J_n = \int_{-1}^{+1} |y - x|^{-\alpha} (1 - y)^{\beta} (1 + y)^{\gamma} T_n(y)\, dy,$$

$$\alpha < 1, \qquad \beta, \gamma > -1, \qquad |x| < 1.$$

Let

$$K = \int_{-1}^{+1} |y - x|^{-\alpha} (y - x)(1 - y)^{\beta} (1 + y)^{\gamma+1} T_n(y)\, dy.$$

Since

$$\begin{aligned}(y - x)(1 + y)\, T_n(y) = {} & (1/4)[T_{n+2}(y) + 2T_n(y) + T_{n-2}(y)] \\ & + ((1 - x)/2)(T_{n+1}(y) + T_{n-1}(y)) - xT_n(y),\end{aligned}$$

we have

$$K = (1/4)(J_{n+2} + 2J_n + J_{n-2}) + ((1 - x)/2)[J_{n+1} + J_{n-1} - xJ_n] \qquad (38)$$

On the other hand, by integration by parts we obtain

$$K = \frac{1-\alpha}{\beta+1}\int_{-1}^{+1} |y-x|^{-\alpha}(1-y)^{\beta+1}(1+y)^{\gamma+1}T_n(y)\,dy$$
$$+\frac{\gamma+1}{\beta+1}\int_{-1}^{+1} |y-x|^{-\alpha}(y-x)(1+y)^{\gamma}(1-y)^{\beta+1}T_n(y)\,dy$$
$$+\frac{1}{\beta+1}\int_{-1}^{+1} |y-x|^{-\alpha}(y-x)(1+y)^{\gamma}(1-y)^{\beta}(1-y^2)\,T_n'(y)\,dy.$$

Since $(1-y^2)\,T_n'(y) = (n/2)[T_{n-1}(y) - T_{n+1}(y)]$ we have

$$K = \frac{1-\alpha}{\beta+1}\left[\frac{J_n}{2} - \frac{1}{4}(J_{n+2}+J_{n-2})\right]$$
$$+\frac{\gamma+1}{\beta+1}\left[-\frac{1}{4}(J_{n+2}+2J_n+J_{n-2}) + \frac{1+x}{2}(J_{n+1}+J_{n-1}) - xJ_n\right]$$
$$+\frac{n}{2(\beta+1)}\left[\frac{1}{2}(J_{n-2}+J_{n+2}) + a(J_{n+1}-J_{n-1})\right]. \tag{39}$$

Equating (38) and (39) yields the desired recurrence relation (15).

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